Fuzzy transition region in a one-dimensional coupled-stable-map lattice

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A coupled-map lattice showing complex behavior in the presence of a fully negative Lyapunov spectrum is considered. A dynamical phase transition from ''frozen'' disorder to chaoticlike evolution upon changing diffusive coupling is studied in detail. Various indicators provide a coherent description of the scenario: the existence of a finite transition region characterized by an irregular alternancy of periodic and chaotic evolution. [S1063-651X(98)07203-1]

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I. INTRODUCTION

Very often the approach to space-time chaos in spatially extended systems is based on the extension of concepts and tools developed for finite- (low-) dimensional systems. For instance, dynamical indicators such as the Kaplan-Yorke dimension and the Kolmogorov-Sinai entropy [1] have been turned into the corresponding intensive indicators, i.e., dimension and entropy *densities* [2]. This strategy is essentially motivated by the hypothesis that the dynamics of chaotic extended systems can be viewed as that of many, almost independent, finite-dimensional subsystems. Although the existence of a limit Lyapunov spectrum [3] provides strong support for such an idea, it is still rather unclear in which sense the evolution of different pieces of, say, a chain of maps is truly uncorrelated.

Even more important is the observation that the infinite dimensionality of the phase space can give rise to entirely new features the understanding of which requires different tools and perhaps will open new perspectives. One such example that will be thoroughly studied in the present paper is the occurrence of chaotic evolution in the presence of a negative maximum Lyapunov exponent. This is indeed a phenomenon that can exist only in an infinite-dimensional phase space, as can be shown with a simple argument based on a *reductio ad absurdum*. An aperiodic evolution requires that the limit set of a generic trajectory contains infinitely many points. If the evolution is confined to a bounded region, there must be at least one accumulation point. Now, since a sufficiently small box centered around any accumulation point contracts in all directions (the maximum Lyapunov exponent being negative), all trajectories in the vicinity of the accumulation point are asymptotically indistinguishable and there can be at most a periodic cycle. This argument breaks down if we have to consider an infinite-dimensional phase space, since in this case the limit set can well be made of infinitely many points, all within a bounded region and yet a finite distance from one another. The above is not only a theoretical possibility, but a feature actually observed in several models such as coupled maps [4–6] and oscillators [7], although no detailed explanation of the underlying mechanisms has yet been provided. What is most striking about this phenomenon is the empirical evidence that the evolution of large enough systems appears to be irregular and stationary in time, so that it makes sense to speak of a ''Lyapunov stable'' chaotic regime. In the following, we shall use the shorthand notation *stable chaos* (SC) to identify this type of behavior.

In a previous paper it has been shown that SC is a robust phenomenon in the sense that it persists in finite regions of the parameter space [5] and it survives even if the discontinuities in the dynamical equations are removed [8,9]. Also the discreteness of the time variable does not seem to be a severe limitation as SC has been observed also in a chain of coupled Duffing oscillators [7]. The only true limitation seems to be the need for a synchronous external forcing of the oscillators.

SC can be partly understood by unveiling the analogy with actual simulations of chaotic maps on digital computers. Any computer has a finite accuracy which is determined by the number of bits used in the internal representation of a real number. As a consequence, even a chaotic map sooner or later must yield a periodic orbit. This apparent limitation has not prevented an effective use of digital computers in the study of deterministic chaos. In fact, if the computer word is sufficiently long, the time required for observing the collapse onto a periodic cycle is so long that this ''transient'' regime is almost indistinguishable from the truly stationary regime of the chaotic map. If one substitutes the length of the computer word with the spatial length of a SC system, we can rephrase the above arguments and thus provide indirect support for the existence of a stationary chaotic regime in infinitely extended systems. However, it is honest to recognize that in the case of deterministic chaos there exists a well developed theory [1] which, starting from Smale horseshoe and Anosov systems, predicts the occurrence of irregular behavior in mappings over the real numbers. In this case, one is faced only with the problem of explaining why an actual simulation reproduces almost exactly the theoretical expectations. Conversely, for what concerns SC, there is mainly numerical evidence and no theory stating that under some specific circumstances one can expect a chaotic evolution in an infinitely extended system. The only pieces of a theory can be constructed at the expense of a further simplification

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which is, however, very enlightening. If one discretizes the continuous state variable in a chain of maps, it is very natural to invoke an analogy with deterministic cellular automata (DCA). In fact, DCAs too can exhibit chaotic behavior only in the infinite-size limit, the finiteness of the number of possible states (2^L for binary automata) necessarily implying the eventual convergence to some periodic orbit. The correspondence between chaotic DCAs and SC can be put on more rigorous ground by first encoding the patterns originated by a SC regime and then trying to reproduce them by some DCAs with a suitable range of interaction. The first step is nothing but an implicit statement regarding the existence of a generating partition. The work done in low-dimensional systems has shown that rather coarse partitions can be constructed which reproduce the dynamics of chaotic maps without loss of information [10,11]. Thus we do not expect this step to be particularly harmful in the context of SC, the only possible problem being the actual construction of a generating partition in specific cases.

The second step is not obvious at all, since it is not known to what extent a pattern with no local production of information can be reduced to a DCA. Let us start the discussion of this issue by recalling that low-dimensional chaotic systems, such as the Hènon map, are equivalent to probabilistic automata, where the probability of the next symbol effectively depends on some previous symbols (their number corresponding to the order of the Markov process). The probabilistic character of the automaton is intrinsically related to the existence of an expanding direction and to the corresponding amplification of uncertainty. The coupling of chaotic maps, as it occurs in spatially extended systems, leads naturally to probabilistic *cellular* automata: the probability of a symbol in a given place at a given time depends not only on the past symbols in the same site as in the previous case, but also on the past symbols in the neighboring sites.

In the case of SC, there is no local amplification of uncertainty, so that it is tempting to conjecture that the future symbol is exactly determined, once the past history of all previous symbols is known. This hypothesis has already been tested in several cases, finding that it would be more appropriate to state that it is the whole new configuration to be predicted with almost no uncertainty. However, in some cases, it has been found that a DCA with a long enough space-time memory suffices to reproduce exactly the observed pattern, while in other cases, the uncertainty of each forecasted symbol decreases and presumably goes to zero only in the limit of an infinite range of interaction [7]. Therefore, it is definitely reasonable to affirm that DCAs represent a subclass of SC systems and what is known about DCAs can be automatically translated into the language of SC. Leaving aside the question of whether SC encompasses some type of behavior absent in DCAs [12,13,7], here we want to stress the important advantage of SC over DCA: the existence of a tunable continuous control parameter. Such a possibility is particularly appealing in view of the conjectured existence of ''complex'' behavior at the edge of chaos [14]. In fact, it has been suggested that a true richness of behavior is observed whenever the underlying rule of a DCA is in some sense halfway between ordered and chaotic rules. However, testing of the above idea requires a parametrization of all the rules and it is not obvious *a priori* what number of parameters must or can be used: there are infinitely many ways to construct automata upon increasing either the interaction range or the number of symbols. Accordingly, some shortcut is attempted in the hope of catching the main features of the organization of all rules in the space of DCAs. The most common approach consists in classifying DCAs according to the so-called activity parameter, i.e., the fraction of local configurations that are mapped onto the same specific symbol (see, e.g., [14–16]); a more refined classification has been proposed by determining several parameters which result from various Markov approximation schemes [17]. However, both approaches suffer the same problem: (i) a continuous tuning of the parameters requires dealing with an infinite number of rules and this can be done in infinitely many ways which, *a priori*, are not equivalent. Moreover, in the former case it is not even clear that different DCAs characterized by the same parameter do behave in the same manner, i.e., that the parametrization is meaningful. In the context of SC, continuous parameters are naturally present in the original model, thus allowing one to study the very same question in a natural and unambiguous way.

In fact, the question of how we pass from a periodic to a chaotic regime in SC systems is perfectly legitimate, as revealed by simulations performed for different choices of the control parameter which show both chaotic and periodic evolution. Accordingly, one can hope to shed light on the transition between these two regimes: is that a standard thermodynamic phase transition, or do we find the signature of ''complexity''? Or is it even as simple as a ''bifurcation''?

The order-to-chaos transition suggests also a comparison with standard space-time intermittency (STI) occurring in chaotic systems [18]. The latter phenomenon has been shown to be strictly related to directed percolation transition. *A posteriori*, this is not very surprising since, on the one hand, a locally chaotic evolution is reminiscent of probabilistic cellular automata (see above), while the ordered dynamical configuration can play the role of an absorbing state. However, the analogy has been shown not to be a complete equivalence between the two phenomena, since finite regions characterized by chaotic behavior cannot be assimilated to truly stochastic domains. A reminiscence of the alternancy of regular and irregular behaviors—typical of low-dimensional systems—indeed survives, leading to a more ''exotic'' evolution on the ordered side of the phase transition [19]. Now, an order-to-chaos transition occurring in a SC system should exhibit even more striking deviations from a percolation transition, basically because any finite ''chaotic domain'' cannot be chaotic at all. This is a first indication that the transition cannot be a ''simple'' equilibrium phase transition, as the studies described in this paper will confirm. However, the link with STI is more subtle than one could naively think. It was already shown that STI can be effectively described by a sequence of DCA constructed by suitable discretization of the local dynamics [20]. In fact, any DCA can be seen as a stepwise map: the smaller is the separation between consecutive steps, the more accurate is the reproduction of the dynamics. Upon changing the control parameter, one passes discontinuously from one to another rule. Thus, for any finite resolution, in a finite number of steps (changes of rule), one passes from ordered to chaotic behavior: no truly continuous parametrization is recovered unless the limit of infinitely many symbols is taken, i.e., the continuous nature of the local variable is restored. However, given any stepwise representation of the local dynamics, one could proceed in a different manner, tilting each of the steps of the local function. In this way, the continuous nature of the variable is immediately restored and if the slopes of the various steps are not too large, the maximum Lyapunov exponent is bounded to be negative. Qualitatively speaking, we have the same phenomenon as in STI, a transition from ordered to chaotic behavior. Quantitatively, in this paper we conjecture that a ''complex'' region is expected to arise in parameter space of SC systems.

Finally, it is worth mentioning another similar transition, extensively studied in the context of neural and Kauffman networks [21]. There, the state variable is discrete (typically binary) and the evolution rule is entirely deterministic exactly as for DCAs. At variance with DCAs, there are (i) quenched disorder: the updating rule operating in a given cell (synapsis) is randomly chosen; and (ii) lack of topology: each cell interacts (is connected) with a randomly chosen set of other cells. In such a context, it has been found that upon decreasing, e.g., the correlation between synaptic couplings (a continuous parameter in the thermodynamic limit), a transition occurs from a chaotic regime to frozen patterns. The transition appears to be a ''standard'' continuous order-to-chaos transition located at a specific value of the control parameter. Within the paradigm of a meaningful complexity occurring at the edge of chaos, it has been conjectured that the most meaningful choice of the parameters for the network to be a realistic model of either gene regulation or neural activity is close to criticality [22].

Here, we investigate a similar order-to-chaos transition occurring in a one-dimensional (1D) lattice of stable maps. Upon varying a control parameter, the system passes from a frozen disorder phase (FDP), characterized by a time-periodic but spatially disordered evolution, to a chaotic phase (CP). The main difference with the behavior of neural networks is that, here, the frozen patterns arise spontaneously notwithstanding the absence of any disorder in the updating rule. The most important result of our investigations is that the transition region is rather intricate, consisting of an irregular alternancy of periodic and chaotic evolutions.

A somehow similar phenomenon has already been investigated in a 2D lattice of stable maps, finding correspondence with a nonequilibrium transition from weak to strong turbulence [23]. In such a case, it has been possible to reproduce the key features of the entire phenomenon by means of a suitable stochastic equation [24]. We suspect that such a transition is indeed close to a true stochastic process since, even in the most ordered (weak turbulence) regime, there are infinitely long interfaces which, in spite of the local stability, can be characterized by a pseudorandom evolution. In fact, it is the infinite dimensionality of the phase space that makes possible the generation of an irregular behavior over an infinite time lag.

This paper is arranged as follows. In Sec. II we present the model, recalling the features of SC and giving a brief overview of the phenomenology occurring for various values of the coupling strength (our control parameter). In Sec. III we study space-time correlation functions and perform damage-spreading analysis since they both allow us to identify a proper order parameter for the transition. In Sec. IV, information-theoretic concepts are introduced for a supplementary investigation. Section V is finally devoted to discussions and conclusions.

II. A MODEL OF STABLE CHAOS

The dynamical system considered in this paper is a one-dimensional lattice of diffusively coupled maps

$$x_i(t+1)=(1-2\varepsilon)f(x_i(t))+\varepsilon[f(x_{i-1}(t))+f(x_{i+1}(t))], \tag{1}$$

where $\varepsilon\in[0,1/2]$ is the coupling constant and periodic boundary conditions are assumed over a length L. The local mapping has the form

$$f(x)=\begin{cases} bx, & 0<x<1/b \\ a+c(x-1/b), & 1/b<x<1. \end{cases} \tag{2}$$

One can easily realize that this mapping can yield stable periodic dynamics for sufficiently small values of c. In what follows, we fix the set of parameter values $\{a=0.07, b=2.70, c=0.10\}$ in such a way that, for any initial condition $x\neq 0$, the attractor of the local mapping is a stable period-3 orbit.

It is worth stressing that the stability of local dynamics (2) implies the stability of map (1), whose maximum Lyapunov exponent turns out to be negative for any value of ε. As a consequence, the long-time evolution of the diffusively coupled system is confined to a periodic attractor. Despite this constraint, we are going to show that very different dynamical regimes can be observed, depending on the coupling ε.

A space-time representation of the evolution can be obtained by encoding the variable x with suitable gray levels. Some typical patterns of the different regimes are reported in Fig. 1. In some cases [see, e.g., Figs. 1(a) and 1(c)], the resulting pattern is basically a random arrangement of different ''stripes,'' each stripe corresponding to a periodic dynamics (frozen disorder); in other cases, there is no evidence of either spatial or temporal order [see, e.g., Figs. 1(b) and 1(d)].

The global properties of a given pattern are governed by the behavior of the domain walls separating different time-periodic phases: if the domain walls do not move, then it is natural to expect a random arrangement of variable-size periodic regions. Alternatively, one may have a ''gas'' of domain walls moving with different velocities and giving rise to a chaotic evolution. The mutual scattering rules between different domain walls thereby determine the properties of the chaotic phase. Even the reader vaguely acquainted with the dynamics of DCA should have recognized in the above sketched regimes the various classes of such models [25]. Therefore it is definitely tempting to use model (1) for checking the existence of a complex phase separating ordered from chaotic motion.

As has already been discussed in Refs. [26,5,27] a criterion for distinguishing chaotic from ordered behavior is provided by the scaling properties of the transient duration with the chain length, starting from random initial conditions. No-

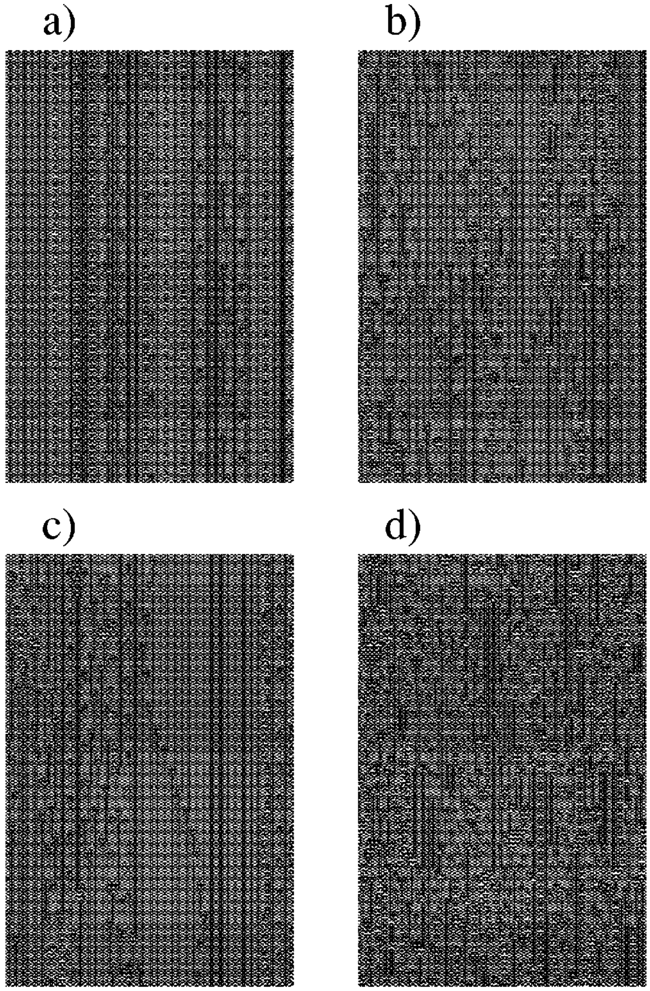

FIG. 1. Four space-time patterns generated by the coupled-map lattice (1) for different values of the coupling constant ε. In all cases, time flows downwards and the patterns, 200×300 wide, are extracted from the evolution of a 3000-site lattice with the same randomly chosen initial condition. Case (a) displays an ordered regime for $\varepsilon = 0.2998$; the more complex pattern in (b) is obtained for $\varepsilon = 0.3004$; (c) displays quasiordered pattern generated for $\varepsilon = 0.3005$; a totally disordered regime is shown in (d) for $\varepsilon = 0.304$.

tice that this approach is the same adopted in the characterization of the order-to-chaos transition occurring, for instance, in neural networks [28]. The transient duration is defined as the number of iterations necessary to observe the first *recurrence,*

$$T_r(L) = \min\{t | d(\{x\}_t, \{x\}_\tau) < \delta, \tau < t\}, \tag{3}$$

where $d(\{x\}_t, \{x\}_\tau)$ is the distance between the configurations at time t and τ, respectively, computed using some specific norm (here we considered the maximum norm). All the conclusions hereafter reported are independent of the actual value of the parameter δ, provided it is small enough (δ has been fixed to 10^{-4} in all our simulations). As was already noticed in Ref. [5], the chaotic regime can be identified by the exponential growth with L of $\langle T_r(L) \rangle$, where the average $\langle \ \rangle$ is performed over the ensemble of random initial conditions.

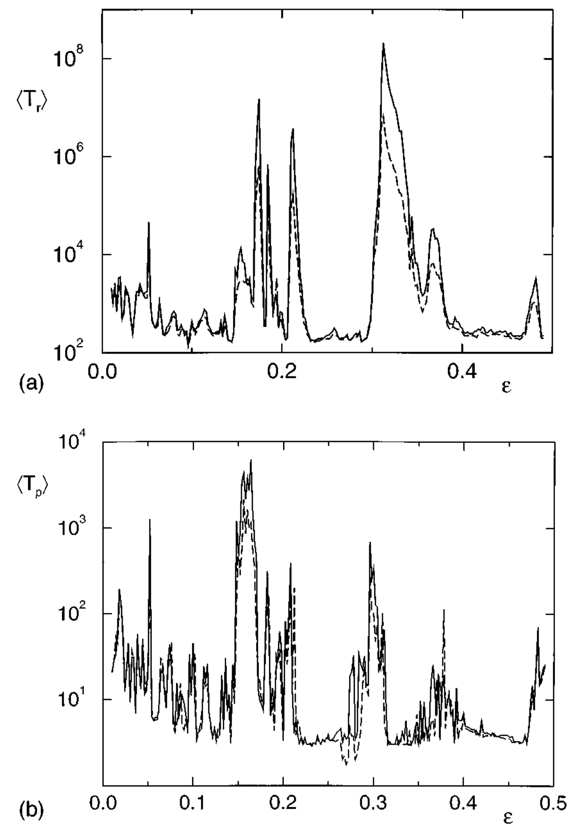

FIG. 2. Average transient time $\langle T_r(L) \rangle$ (a) and average period $\langle T_p(L) \rangle$ (b) versus ε. Both averages are performed over 200 random initial conditions. Solid and dashed curves refer to a chain length $L = 50$, 40, respectively.

A global picture of the average transient time $\langle T_r(L) \rangle$ is shown in Fig. 2(a) for different values of ε. The strong variations in the order of magnitude of $\langle T_r(L) \rangle$ do confirm the visual impression of an irregular alternancy of ordered and chaotic regimes. This is further strengthened by the comparison between the solid and the dashed curves (corresponding to $L = 50$ and 40, respectively), that single out the chaotic regions as those where the solid curve is consistently above the dashed one.

Before discussing the various approaches used for investigating the transition region, let us comment about another aspect of the evolution of finite chains: the period $T_p(L)$ of the asymptotic state. In principle, $T_p(L) < T_r(L)$; in practice, $T_p(L)$ can be much shorter, as seen in Fig. 2(b), where the average period $\langle T_p(L) \rangle$ is reported versus ε, showing strong fluctuations, while it may remain rather ''short'' deeply inside the chaotic regions. It is worth stressing that this phenomenology is completely different from that observed in neural networks, where the chaotic phase is characterized also by periods as long as transients [28]. Nonetheless, in model (1), one observes an accumulation of longer and longer periods when any transition is approached from the ''ordered'' side.

III. CHARACTERIZATION OF THE PHASE TRANSITION

Direct inspection of Fig. 2 shows that the widest chaotic region is approximately located in the interval $\varepsilon \in [0.3, 0.4]$. Incidentally, it is in this region that the first evidence of SC was found in this model for $\varepsilon = 1/3$ (see Ref.

[5]). For this reason we have chosen to point our attention to the parameter region close to $\varepsilon = 0.3$. Transient analysis and spatiotemporal patterns obtained from the simulations give clear evidence that, sufficiently below (above) $\varepsilon = 0.3$, there is a whole range of ε values, where frozen disorder (chaotic dynamics) takes place. In other words, we are in the presence of a phase transition between different dynamical regimes. In what follows we shall characterize it by analyzing the behavior of some observables, aiming also to identify an order parameter. More precisely, in the first subsection, we discuss the properties of the spatiotemporal correlation functions, finding that only the CP displays a temporal decay to zero. In the second subsection, we study the propagation of initially localized perturbations (damage-spreading analysis), that is found to drop to zero in the FDP. A careful application of the above tools has consistently revealed that there is not a single threshold separating the two phases, but rather a whole ''fuzzy'' region $\varepsilon \in [0.3, 0.3005]$, where periodic and chaotic behaviors alternate in an apparently irregular manner.

We believe that the peculiarity of this transition should be attributed not only to the deterministic nature of the model (as in the case of STI), but also to the specific absence of a local source of chaos.

A. Correlation functions

Space-time correlation functions are common tools for describing the statistical properties of the motion in spatially extended systems. In fact, they provide a first quantitative criterion apt to classify the various regimes observed in the dynamics of model (1). In particular, they allow one to check whether the phase transition can be associated with the appearance of spatial long-range order.

The spatiotemporal correlation function is defined as

$$C(i,j;t,\tau) = \langle x_i(t) x_{i+j}(t+\tau) \rangle_t - \langle x_i(t) \rangle^2, \qquad (4)$$

where the average $\langle \; \rangle$ is performed over initial conditions made of independent, identically and uniformly distributed random variables $x_i(0)$.

In view of the periodic boundary conditions and because of the translational invariance of the initial conditions, $C(i,j;t,\tau)$ does not depend on i. Conversely, for what concerns the dependence on t, there is no reason *a priori* for it to be irrelevant. On the other hand, if the system approaches a stationary regime for sufficiently large t, this dependence is practically negligible. The only case where this does not occur is FDP, when the phase of the time periodicity still plays a role. As in the present investigation such a dependence is not relevant, from here on we drop the dependence on t in Eq. (4).

Moreover, since numerical simulations show the absence of traveling structures in the asymptotic configurations, here we can consider separately the spatial and temporal behavior of Eq. (4).

Upon these remarks, we define the spatial correlation function as

$$C_S(j) = \langle x_i(t) x_{i+j}(t) \rangle_t - \langle x_i(t) \rangle_t^2, \qquad (5)$$

where the subscript t indicates that the average is performed also over time. Independently of ε, $C_S(j)$ exhibits an exponential decay over a few lattice units (see Fig. 3), indicating that, presumably, spatial long-range order does not occur at the transition. This is compatible with the qualitative picture suggested by the patterns, which all present spatial disorder. This behavior makes doubtful the perspective of considering the present phase transition as a nonequilibrium critical phenomenon.

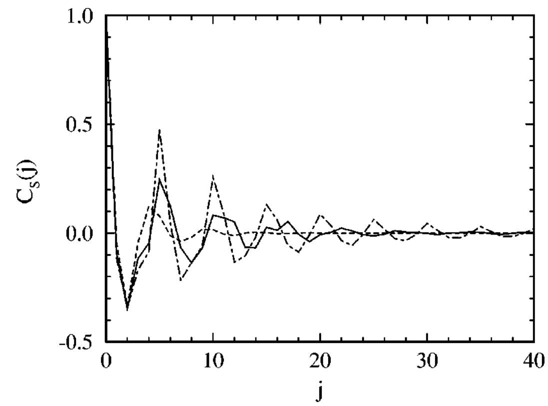

FIG. 3. Spatial correlation function (5) for different values of ε: 0.2998 (full line), 0.3005 (dashed line), 0.3040 (dot-dashed line). All the curves are obtained by averaging over 500 random initial conditions and 10^5 time steps.

The evolution of model (1) can be characterized by the behavior of the temporal correlation function

$$C_T(\tau) = \langle x_i(t) x_i(t+\tau) \rangle_t - \langle x_i(t) \rangle_t^2. \qquad (6)$$

In order to improve the statistics, in the numerical simulations we have also performed an average over lattice sites separated by a distance larger than the spatial correlation length.

At variance with $C_S(j)$, $C_T(\tau)$ shows a very sensitive dependence on ε, especially when approaching the fuzzy region, thus confirming the existence of different phases in the evolution of the system. The scenario can be summarized as follows. In the whole range of ε values that we have considered, $C_T(\tau)$ displays period-2 oscillations that appear to originate from the abundancy of stable period-2 orbits. For ε below the lower ''threshold'' $\varepsilon_c(1) = 0.3$, $C_T(\tau)$ does not decay [see Fig. 4(a)]. This confirms the presence of a well defined phase, corresponding to a time-periodic but spatially disordered dynamics (FDP).

For values of ε inside the range $[\varepsilon_c(1), \varepsilon_c(2)]$, $C_T(\tau)$ may decay either to zero, as in CP (see below), or towards a nonzero asymptotic value [Fig. 4(b)], indicating the presence of (temporal) long-range order.

For $\varepsilon \geqslant \varepsilon_c(2) = 0.3005$, $C_T(\tau)$ decays to zero [see, for instance, Fig. 4(d)]. A good observable for the characterization of the transition from CP to FDP is represented by the correlation time θ as determined from the asymptotic decay of the temporal correlation function,

$$\mathrm{Env}\{C_T(\tau)\} \sim \exp(-\tau/\theta). \qquad (7)$$

In fact, θ becomes very large as soon as ε approaches $\varepsilon_c(2)$ from above, as can be seen in Table I. In this perspective, θ is an appropriate order parameter, being finite in CP and consistently equal to ∞ in the FDP. The delicacy of the nu-

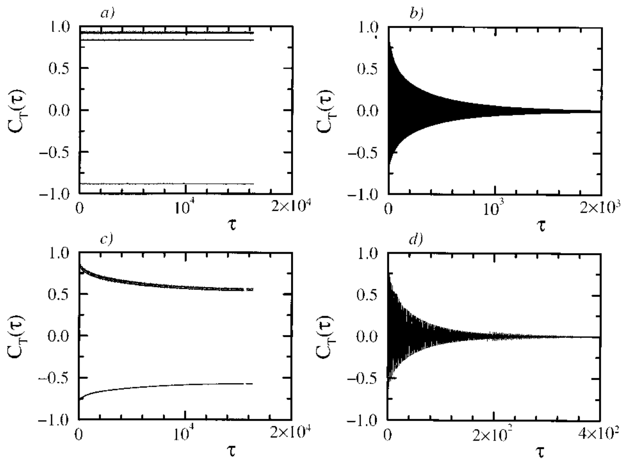

FIG. 4. Temporal-correlation functions for the same values of ε considered in Fig. 1. For the sake of clarity, in (a) and (c), the points are not connected by lines. In the ordered regime (a) the correlation function does not decay and exhibits essentially period-2 oscillations; in the complex pattern (b), a slow decay to zero is observed; in the ordered regime (c), inside the fuzzy region, period-2 oscillations coexist with a temporal decay to a finite asymptotic value; in the chaotic region (d), period-2 oscillations modulate a much faster decay than in case (b).

merical simulations is such that we cannot say anything about the possible existence of a critical behavior, when the upper border $\varepsilon_c(2)$ of the fuzzy region is approached.

B. Damage-spreading analysis

The irregular dynamics observed in our model is produced by transport rather than by local production of information, which is not present in view of a negative maximum Lyapunov exponent. The similarity with DCA, discussed in the Introduction, suggests that the main features of this transport mechanism can be analyzed by studying the propagation of finite disturbances. In DCA language, this is the so-called damage-spreading analysis [25]. In practice, it amounts to determine the effects produced on the pattern by localized perturbations of the initial state. An unbounded growth of the region affected by the perturbation is usually considered as an indication of a chaotic evolution [25]. In fact, this means that disturbances arising at the boundaries can travel undamped through the whole system.

A perturbation can be introduced in the following way. Let $\mathcal{X}_1=\{x_1,x_2,\ldots,x_L\}$ and $\mathcal{X}_2=\{y_1,y_2,\ldots,y_L\}$ represent two initial configurations such that

TABLE I. The correlation time θ for various ε values inside the chaotic phase.

ε	θ
0.301	(3300 ± 200)
0.302	(310 ± 80)
0.304	(160 ± 40)
0.306	(230 ± 40)
0.308	(53 ± 10)
0.310	(19 ± 2)

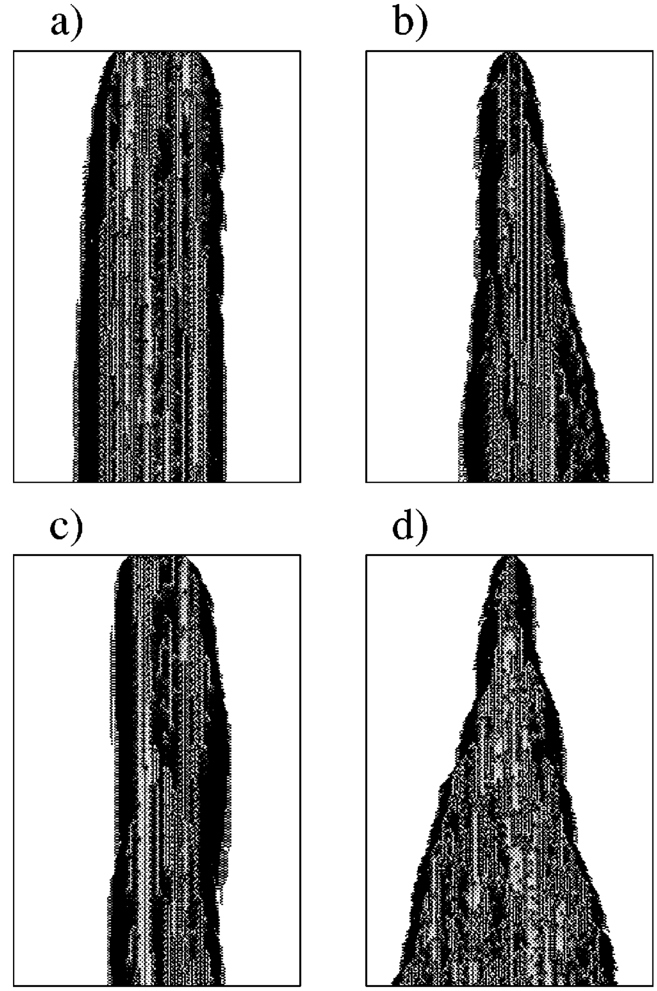

FIG. 5. Damage-spreading patterns corresponding to the same ε values considered in Fig. 1.

$$y_i=\begin{cases} x_i+\delta_i, & |i-L/2|\leq S \\ x_i & \text{elsewhere,} \end{cases}$$

where $\delta_i\sim O(1)$, such that nonlinearities can play an effective role (the only nonlinearity present in our model is the discontinuity in the map); L is the chain length, and S is the size of the region where the two configurations are initially different. Then, $\mathcal{X}_2$ is said to be a perturbation of $\mathcal{X}_1$. Typical damage-spreading patterns close to and inside the transition region are shown in Fig. 5. Direct inspection indicates that an effective spreading occurs in CP [see Fig. 5(d)].

The transmission rate of information is then measured as the average velocity of increase of the perturbation size. In practice this can be defined by making reference to two different quantities: (a) the position of the perturbation front; and (b) the distance between two configurations. In case (a) one first defines the left and the right fronts of the perturbation

$$F_l(t)=\min\{1\leq i\leq L:|x_i(t)-y_i(t)|>0\},$$

$$F_r(t)=\max\{1\leq i\leq L:|x_i(t)-y_i(t)|>0\}. \tag{8}$$

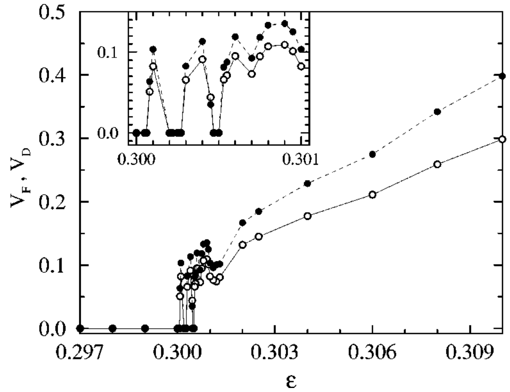

FIG. 6. The damage-spreading velocities V_F and V_D vs ε. The inset amplifies the fuzzy region $[\varepsilon_c(1),\varepsilon_c(2)]$, where periodic and chaotic regimes irregularly alternate.

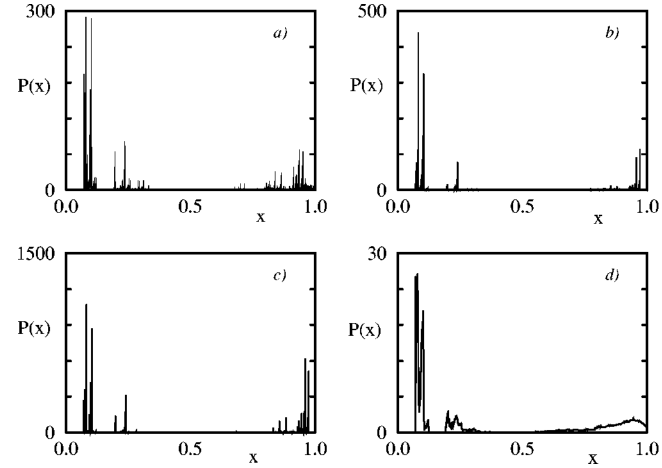

FIG. 7. Probability density $P(x)$ of the site variable for the same ε values considered in Fig. 1.

Insofar as the left-right spatial symmetry is not broken, as in the present model, the two definitions are equivalent. The corresponding front velocity is therefore

$$V_F = \lim_{t\to\infty} \frac{\langle F_{l,r}(t)\rangle}{t}. \tag{9}$$

In case (b) the distance D between two configurations of the system is given by

$$D(t) = \frac{1}{L}\sum_{i=1}^{L} |x_i(t) - y_i(t)| \tag{10}$$

(notice that D is a straightforward generalization of the Hamming distance usually adopted in the study of DCA behavior). Accordingly, the corresponding average damage-spreading velocity reads

$$V_D = \lim_{t\to\infty} \frac{\langle D(t)\rangle}{t}. \tag{11}$$

In both the above definitions, $\langle\ \rangle$ denotes the average performed over initial conditions. Positive values of V_F and V_D indicate that any two nearby configurations tend to separate in time. In this sense, damage-spreading analysis can be considered analogous to the Lyapunov stability analysis.

In numerical simulations we have averaged over 500 initial conditions in order to reduce fluctuations. The results reported in Fig. 6 show that V_F and V_D are equivalent modulo a scale factor. This implies that the damage process acting inside the propagation cone is uncorrelated with the front dynamics. The resulting scenario is the same as the one suggested by the correlation-function analysis: for $\varepsilon < \varepsilon_c(1)$ $[\varepsilon > \varepsilon_c(2)]$ both V_F and V_D are zero (nonzero). This confirms that damage-spreading velocities are reliable order parameters to distinguish between FDP and CP.

Inside the fuzzy region $[\varepsilon_c(1),\varepsilon_c(2)]$ both zero- and nonzero-velocity regimes finely alternate without any apparent regularity. We want to stress that velocity fluctuations clearly visible in the inset of Fig. 6 are not an artifact following from either statistical uncertainty or finite-size effect. In fact, the initial conditions have been taken after discarding a sufficiently long transient, which, in some cases, amount to more than 20 000 iterations. In particular, the transient has been estimated from the relaxation properties of the ensemble average of the variables x_i. Moreover, we increased the system size until we found evidence that the velocity does not depend on L. This means that in some ''critical'' cases we had to work with lattices of 6000 sites. Finally, the simulations have been allowed to evolve for a sufficiently long time (up to 10^5 iterations) to accurately determine the asymptotic behavior.

It is interesting to note that in those ambiguous cases where $C_T(\tau)$ was apparently decaying to a finite value V_F and V_D are strictly zero. This confirms the conjecture that these are truly ordered regimes.

IV. INFORMATION-THEORETIC ANALYSIS

The different features of the patterns in the fuzzy region indicate that the underlying dynamical mechanism is associated to a sequence of structural changes of the phase space. A proper method for quantifying this scenario amounts to studying the probability distribution function of the state variable x_i's,

$$P(x) = \int_0^1 dx_1 \cdots \int_0^1 dx_L \rho(x_1, \ldots, x_L)\ \delta(x - x_1), \tag{12}$$

where $\rho(x_1, \ldots, x_L)$ represents the unknown invariant measure generated under the evolution law (1). If one assumes that ρ is defined by averaging over the usual ensemble of initial conditions, then translation invariance is automatically ensured and the choice of the non-dummy variable (here x_1) is irrelevant. The histograms of $P(x)$, shown in Fig. 7, have been obtained by averaging over 500 initial conditions and over the whole time span of the simulations (obviously, after discarding a suitable transient). In CP, where the damage-spreading velocities are strictly positive, $P(x)$ exhibits a peaked distribution superposed to a continuous component [see, e.g., Fig. 7(d)]. The peaks are located in correspondence to the values of some space-time periodic orbits, which still turn out to play a role even in the chaotic regime. Below $\varepsilon_c(2)$ the continuous component is negligible and

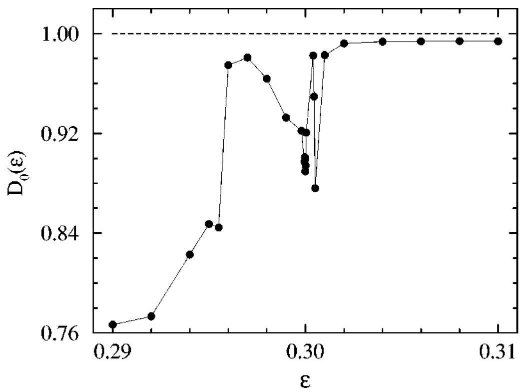

FIG. 8. Information dimension D_0 of the single variable invariant measure vs ε.

only the peaked structure survives [see Figs. 7(a)–7(c)]. This occurs irrespectively of the velocity, that may be either very small [case (b)] or strictly zero [case (c)]. The effective absence of a continuous component in the fuzzy region, even if V_F and V_D are positive, leaves open the question whether this is due to an insufficient spatial resolution.

More refined information can be obtained by partitioning the unit interval into subintervals of equal length Δ and thereby computing the entropy

$$S(\Delta) = -\sum_i \mu_i \ln \mu_i, \tag{13}$$

where μ_i is the integral of $P(x)$ over the ith subinterval. The scaling behavior of $S(\Delta)$ yields the information dimension

$$D_0 = \lim_{\Delta \to 0} -\frac{S(\Delta)}{\ln \Delta}, \tag{14}$$

which is a natural indicator quantifying the strength of the apparent singularities. In CP, D_0 is steadily close to 1, confirming that on sufficiently fine scales the distribution is continuous (the distance from 1 is, indeed, not appreciable). In the fuzzy transition region, D_0 exhibits irregular oscillations, while below $\varepsilon_c(1)$ one observes a smooth tip followed by a sharp decay (see Fig. 8). Accordingly, in FDP, D_0 cannot be considered as a meaningful order parameter since the information dimension can be as large as in CP. Note that the origin of a strictly positive dimension D_0, even in FDP, stems from the irregular spatial structure irregular alternancy of periodic stripes. In fact, when the time evolution of the coupled-map lattice (CML) is periodic, one can imagine obtaining the same invariant measure upon iterating in space model (1), after imposing periodic boundary conditions in time. In this perspective, one can conjecture that, at variance with the time evolution, the spatial iteration yields a positive Lyapunov exponent. Accordingly D_0 should be read as the fractal dimension of the corresponding strange repeller. A quantitative verification of these ideas, requiring special care in dealing with the escape rate from the repeller, will be performed elsewhere.

Nonetheless, we have refined our analysis by projecting the invariant measure ρ onto higher-dimensional spaces,

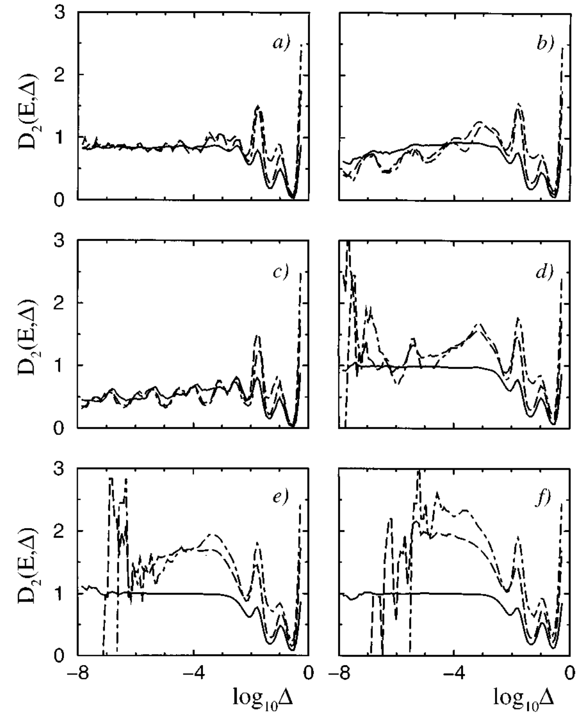

FIG. 9. Effective dimension D_2 vs $\log_{10}\Delta$ for embedding dimension $E=1,2,3$ (solid, dashed, and dot-dashed lines, respectively) for different values of ε: 0.2998 (a), 0.3004 (b), 0.3005 (c), 0.3015 (d), 0.302 (e), and 0.304 (f).

$$P(x_1, \dots, x_E) = \int_0^1 dx_{E+1} \cdots \int_0^1 dx_L \rho(x_1, \dots, x_L), \tag{15}$$

and studying the corresponding fractal dimensions. Notice that for $E=1$, the above equation reduces to Eq. (12). An effective study of $P(x_1, \dots, x_E)$ can be performed by interpreting a spatial configuration as a time series and thereby applying embedding techniques (see, e.g., [2]). We have applied the Grassberger-Procaccia method [29] to configurations of length $M=10^5$. The technique consists in calculating the correlation integral

$$\mathcal{N}(E,\Delta) = \frac{2}{M(M-1)} \sum_{i<j} \Theta(\Delta - \|x_i - x_j\|_E), \tag{16}$$

where $\Theta(t)$ is the Heaviside function and $\| \, \|_E$ is some norm in an E-dimensional space. The estimation of the correlation dimension D_2 can be obtained from the asymptotic behavior of the effective dimension

$$D_2(E,\Delta) = -\frac{\partial \ln \mathcal{N}}{\partial \ln \Delta} \tag{17}$$

as $\Delta \to 0$. The results for different values of ε and embedding dimension $E=1,2,3$ are reported in Fig. 9. At variance with the previous cases, this analysis reveals two well separated regimes. Above $\varepsilon_d = 0.301$, D_2 increases with the embedding dimension [please notice that the apparent decrease of the effective dimension observed in Figs. 9(e) and 9(f) at small distances is an artifact following from a lack of sufficient statistics], while, below ε_d, the fractal dimension is independent of E [see Figs. 9(a)–9(d)], irrespective of whether the evolution is actually periodic or chaotic. This means that, in FDP, the invariant measure has a finite fractal dimension, which can be determined already for $E=1$. This is not so surprising as long as the temporal evolution is periodic; it is less obvious that the same behavior of the fractal dimension is found for the aperiodic regimes occurring inside the fuzzy region. Another question that is left open by the simulations is the actual dimension of the invariant measure deeply inside CP: it is not clear whether the dimension is finite although very large, or if it is infinite as in standard space-time chaos. Unfortunately, at present there are no theoretical arguments that can help clarify this question: in particular, we cannot resort to the Kaplan-Yorke conjecture for concluding that the dimension should be proportional to the system size and thus infinite in the thermodynamic limit.

V. CONCLUSIONS AND PERSPECTIVES

A peculiar dynamical transition between a periodic (FDP) and a chaotic phase (CP) has been identified in a one-dimensional Lyapunov-stable CML model. Both the correlation time θ of $C_T(\tau)$ and the damage-spreading velocities (V_F, V_D) allow one to clearly distinguish between the two phases: FDP is characterized by zero-velocity and long-range time correlations, while a finite V and an exponential decay of the correlations are observed in CP. Both indicators reveal the existence of a finite transition region where either periodic or chaotic evolution arises, depending on the control parameter. This scenario is vaguely reminiscent of the alternancy of periodic windows and chaotic regimes in low-dimensional systems such as the logistic map. However, here the ‘‘bifurcation’’ diagram must be more complex in view of the (infinitely) many periodic dynamics that can be found along the lattice for a given parameter value. Accordingly, it is not even clear whether finite periodic windows must always exist. In any case, numerical analysis alone cannot reveal whether infinitely many transition points are to be found in the fuzzy region.

An important step towards a more detailed comprehension of this phenomenon will be performed when the borders of the transition region will be accurately determined. There, one can hope to find a true critical behavior, such as the period doubling phenomenon in the context of low-dimensional chaos. The difficulty of the task is, however, attested to by the results of the fractal-dimension analysis performed on spatially embedded configurations. This is the only analysis which is blind to the irregular alternancy of periodic and chaotic evolution in the fuzzy region and yet reveals that CP is qualitatively different from FDP. In fact, it is only inside CP that we have found fractal dimensions definitely larger than 1: this suggests that the chaotic evolution observed in the fuzzy region is of a different type from the evolution appearing inside CP. Does this mean that we are in the presence of even more than two phases?

It is very tempting to interpret the fuzzy region as evidence of the complex behavior conjectured to exist at the edge of chaos, at least in the context of deterministic cellular automata. In the past, this question has been approached in two different ways: (i) by sampling the space of DCA rules with a suitable metarule (e.g., the so-called genetic algorithms); (ii) by suitably parametrizing the DCA and thereby studying the resulting sequence of behaviors. Here we are not primarily concerned with the former approach that we have mentioned mainly for the sake of completeness. We limit ourselves to noting that it is related to the idea that life is a process driving a (biological) system to this edge, which is the only place where it is conjectured that computational tasks can be performed.

In the framework of the latter approach, we should mention that a similar scenario has been observed and described in [20], while discussing various DCA approximations of spatiotemporal intermittency, but there it was only observed for finite approximations, but it disappears in the continuous limit. In [16] too, a blurred transition region has been identified while studying the behavior of a large ensemble of DCAs, but one cannot exclude that the fuzziness is a consequence of the parametrization introduced *ad hoc* to study the transition from periodic to chaotic rules. Apparently, it is only in the context of stable chaos that this phase transition conserves a nonstandard character and yet can be meaningfully investigated. However, a more complete characterization of the various regimes is still required before drawing definite conclusions. For the time being, we limit ourselves to conjecture that the absence of local sources of randomness (either due to stochastic terms or to deterministic chaos) is at the root of this complex scenario.

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